Partial Solution Set, Leon §7.4

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7.4.1 Determine $\|\cdot\|_F, \|\cdot\|_\infty$, and $\|\cdot\|_1$ for each of the following matrices.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The norms are $\|A\|_F = \sqrt{2}$, $\|A\|_\infty = \|A\|_1 = 1$.

(d) $= \begin{bmatrix} 0 & 5 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.
The norms are $\|A\|_F = 7$, $\|A\|_\infty = 6$, and $\|A\|_1 = 10$.

7.3.3 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Show that $\|A\|_2 = 1$.

Proof: If $\mathbf{x} = \begin{bmatrix} x_1 \\ x^2 \end{bmatrix} \in \mathbf{R}^2$, then $A\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$, so

$$\|A\|_2 = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\mathbf{x}\neq\mathbf{0}} \frac{|x_1|}{\sqrt{x_1^2 + x_2^2}} \leq 1,$$

with equality when $x_2 = 0$. □

7.4.5 Let $\|\cdot\|_M$ denote a matrix norm on $R^{n\times n}$, and let $\|\cdot\|_v$ denote a vector norm on R^n. Let I denote the $n \times n$ identity matrix. Show the following:

(a) If $\|\cdot\|_M$ and $\|\cdot\|_v$ are compatible, then $\|I\|_M \geq 1$.

(b) If $\|\cdot\|_M$ is subordinate to $\|\cdot\|_v$, then $\|I\|_M = 1$.

Solution:

(a) Suppose $\|\cdot\|$ and $\|\cdot\|$ are compatible. Then for any $A \in R^{n\times n}$ and $\mathbf{x} \in R$,

$$\|A\mathbf{x}\| \leq \|A\|_M \|\mathbf{x}\|_v.$$

It follows that, for any nonzero $\mathbf{x} \in R^n$,

$$\|\mathbf{x}\|_v = \|I\mathbf{x}\|_v \leq \|I\|_M \|\mathbf{x}\|_v,$$

so

$$\|I\|_M \geq \frac{\|\mathbf{x}\|_v}{\|\mathbf{x}\|_v} = 1.$$

(b) Suppose $\|\cdot\|_M$ is subordinate to $\|\cdot\|_v$. Then for any $A \in R^{n\times n}$,

$$\|A\|_M = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|A\mathbf{x}\|_v}{\|\mathbf{x}\|}.$$

In particular,

$$\|I\|_M = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|I\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{x}\|_v}{\|\mathbf{x}\|_v} = 1.$$

7.4.11 Let $\|\cdot\|$ denote the family of vector norms and let $\|\cdot\|_M$ be a subordinate matrix norm. Show that

$$\|A\|_M = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

Solution: Let A be an $m \times n$ matrix, and $\mathbf{0} \neq \mathbf{x} \in \mathbf{R}^n$.

Then

$$\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\frac{1}{\|\mathbf{x}\|}\|A\mathbf{x}\|}{\frac{1}{\|\mathbf{x}\|}\|\mathbf{x}\|} = \|A\mathbf{u}\|,$$

where $\mathbf{u} = \dfrac{\mathbf{x}}{\|\mathbf{x}\|}$.

7.4.12 Let A be an $n\times n$ matrix, and let $\|\cdot\|_M$ be a matrix norm that is compatible with some vector norm on $\mathbf{R}^n$. If λ is an eigenvalue of A, show that $|\lambda| \leq \|A\|_M$.

Proof: Suppose A is $n \times n$, λ is an eigenvalue of A with associated eigenvector $\mathbf{x}$, and that $\|A\|_M$ is compatible with some vector norm $\|\cdot\|_V$. Then

$$|\lambda|\|\mathbf{x}\|_V = \|\lambda\mathbf{x}\|_V = \|A\mathbf{x}\|_V \leq \|A\|_M\|\mathbf{x}\|_V.$$

Since $\|\mathbf{x}\|_V \neq 0$, it follows that $|\lambda| \leq \|A\|_M$. □